

# MODIFIED DOUBLE POISSON BRACKETS

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ABSTRACT. We propose a non-skew-symmetric generalization of the original definition of double Poisson Bracket by M. Van den Bergh. It allows one to explicitly construct more general class of  $H_0$ -Poisson structures on finitely generated associative algebras. We show that modified double Poisson brackets inherit certain major properties of the double Poisson brackets.

## 1. INTRODUCTION

Application of the noncommutative geometry program to symplectic manifolds was originated in [Kon93]. Following general philosophy formulated by M. Kontsevich any algebraic property that makes geometric sense is mapped to its commutative counterpart by the functor  $\text{Rep}_N$ :

$$\text{Rep}_N : \text{fin. gen. Associative algebras} \rightarrow \text{Affine schemes},$$

which assigns to a finitely generated associative algebra  $\mathcal{A}$  a scheme of its'  $N \times N$  matrix representations<sup>1</sup>

$$\text{Rep}_N(\mathcal{A}) = \text{Hom}(\mathcal{A}, \text{Mat}_N(\mathbb{C})).$$

In line with this general philosophy, M. Van den Bergh [dB08] proposed a definition of the double Poisson bracket on associative algebra which induces a conventional Poisson bracket on the coordinate ring of matrix representations.

On the contrary, W. Crawley-Boevey [CB11, CBEG07] suggested related, yet another definition of the noncommutative analogue of the Poisson bracket, the so-called  $H_0$ -Poisson structure. The latter has weaker requirements and in general provides a conventional Poisson bracket only on the moduli space of representations. Double Poisson bracket induces an  $H_0$ -Poisson structure but not vice versa.

One of the major advantages of the double Poisson bracket as opposed to an  $H_0$ -Poisson structure is that for a finitely generated associative algebra it is defined completely by its' action on generators. This allows one to provide numerous explicit examples of double Poisson brackets [PVDW08, BT16] and even carry out certain partial classification problems [ORS13].

In this note we provide an extension of the original ideas of M. Van den Bergh and W. Crawley-Boevey. We define a notion of the *modified* double Poisson bracket (see Definition 1) which allows one to construct explicitly more general examples of  $H_0$ -Poisson brackets on finitely generated algebras. We support our definition with new examples of modified double Poisson brackets in Sec. 4 and calculate corresponding dimensions of symplectic leaves of the induced Poisson structures on the moduli space.

In Section 5 we use the algebra of noncommutative poly-vector fields introduced in [dB08] to construct non-skew-symmetric biderivations. We introduce the notion of a modified double Poisson bivector and present an essential example.

Finally, in Section 6.1 we investigate brackets on representation algebras induced by the modified double Poisson brackets. We show that some recent results of G. Massuyeau and V. Turaev [MT15] can be extended beyond skew-symmetric case as well.

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<sup>1</sup>Throughout the text we assume the ground field to be  $\mathbb{C}$ . All unadorned tensor products, algebras, and schemes are over  $\mathbb{C}$  unless specified otherwise.

## 2. MODIFIED DOUBLE POISSON BRACKET

Let  $\mathcal{A} = \mathbb{C}\langle x^{(1)}, \dots, x^{(k)} \rangle / \mathcal{R}$  be an associative algebra over  $\mathbb{C}$ , which is finitely generated by  $\{x^{(1)}, \dots, x^{(k)}\}$ , possibly with some finite number of relations  $\mathcal{R}$ .

**Definition 1.** A modified double Poisson bracket on  $\mathcal{A}$  is a map  $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$  s.t. for all  $a, b, c \in \mathcal{A}$

- (1a)  $\llbracket a \otimes bc \rrbracket = (b \otimes 1)\llbracket a \otimes c \rrbracket + \llbracket a \otimes b \rrbracket(1 \otimes c)$
- (1b)  $\llbracket ab \otimes c \rrbracket = (1 \otimes a)\llbracket b \otimes c \rrbracket + \llbracket a \otimes c \rrbracket(b \otimes 1)$
- (1c)  $\{a \otimes \{b \otimes c\}\} - \{b \otimes \{a \otimes c\}\} = \{\{a \otimes b\} \otimes c\}$  where  $\{-\} := \mu \circ \llbracket - \rrbracket$
- (1d)  $\{a, b\} + \{b, a\} = 0 \text{ mod } [\mathcal{A}, \mathcal{A}]$

The fact that we do not require skew-symmetry in a sense of Van den Bergh  $\llbracket a, b \rrbracket = -\llbracket b, a \rrbracket^{op}$  is the major difference with the case studied in [dB08, ORS12]. To distinguish with definitions introduced in [dB08] we call this object **Modified** double Poisson Bracket. In Section 4 we show that there are essential examples of the modified double Poisson brackets.

**Corollary 2.** Composition with the multiplication map  $\{-\} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$  defines an  $H_0$ -Poisson structure, namely for all  $a, b, c \in \mathcal{A}$

- (2a)  $\{a, bc\} = b\{a, c\} + \{a, b\}c,$
- (2b)  $\{ab, c\} = \{ba, c\},$
- (2c)  $\{a, \{b, c\}\} - \{b, \{a, c\}\} = \{\{a, b\}, c\},$
- (2d)  $\{a, b\} + \{b, a\} \equiv 0 \text{ mod } [\mathcal{A}, \mathcal{A}].$

In particular, the latter implies that  $\{-\} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$  factors through  $\{-\} : \mathcal{A}/[\mathcal{A}, \mathcal{A}] \otimes \mathcal{A} \rightarrow \mathcal{A}$  which we denote by the same brackets.

**Corollary 3.** An  $H_0$ -Poisson structure  $\{-\}$  in turn induces a Lie Algebra structure  $\{-\}^{Lie} : \mathcal{A}/[\mathcal{A}, \mathcal{A}] \otimes \mathcal{A}/[\mathcal{A}, \mathcal{A}] \rightarrow \mathcal{A}/[\mathcal{A}, \mathcal{A}]$  on abelianization  $\mathcal{A}_{\natural} = \mathcal{A}/[\mathcal{A}, \mathcal{A}]$  of  $\mathcal{A}$ .

## 3. POISSON BRACKETS ON THE MODULI SPACE OF REPRESENTATIONS

Double derivation property introduced to Definition 1 at the same time provides a constructive definition for a certain subclass of  $H_0$ -Poisson structures and allows one to establish a precise correspondence between  $H_0$ -Poisson structures and geometry. Throughout this section we will review main ideas of pioneering papers [CB99, CBEG07, dB08] and apply them to the context of the Modified Double Poisson Bracket.

**3.1. Representation scheme.** As before, let  $\mathcal{A} = \langle x^{(1)}, \dots, x^{(k)} \rangle / \mathcal{R}$  be a finitely generated associative algebra with a finite set of relations  $\mathcal{R}$ . Each representation of  $\mathcal{A}$  in  $Mat_N(\mathbb{C})$  can be defined by the image of the generators, let

$$(3) \quad \varphi(x^{(i)}) = \begin{pmatrix} x_{11}^{(i)} & \dots & x_{1N}^{(i)} \\ \vdots & & \vdots \\ x_{N1}^{(i)} & \dots & x_{NN}^{(i)} \end{pmatrix}.$$

Representations of  $\mathcal{A}$  then form an affine scheme  $\mathcal{V}$  with a coordinate ring  $\mathbb{C}[\mathcal{V}] := \mathbb{C} \left[ x_{j,k}^{(i)} \right] / \varphi(\mathcal{R})$ . Denote as  $\mathbb{C}_{\mathcal{V}}$  — the corresponding sheaf of rational functions. Then  $\varphi : \mathcal{V} \times \mathcal{A} \rightarrow Mat_N(\mathbb{C})$ . For a general point  $m \in \mathcal{V}$  map  $\varphi(m, -)$  provides an  $N$ -dimensional matrix representation of  $\mathcal{A}$ . Hereinafter, we often omit the first argument of  $\varphi$  where it is assumed to be a function on  $\mathcal{V}$ .

**3.2. Moduli space of representations.** There is a natural action of  $GL_N(\mathbb{C}) \curvearrowright Mat_N(\mathbb{C})$  which corresponds to the change of basis in the underlying finite dimensional module. It induces the  $GL_N(\mathbb{C})$  action on the sheaf of rational functions  $\mathbb{C}_{\mathcal{V}}$ . We denote as  $\mathbb{C}[\mathcal{V}]^{inv} \subset \mathbb{C}[\mathcal{V}]$  (respectively  $\mathbb{C}_{\mathcal{V}}^{inv} \subset \mathbb{C}_{\mathcal{V}}$ ) the subalgebra of  $GL_N(\mathbb{C})$  invariant elements. We refer to the orbit of the  $GL_N(\mathbb{C})$  action as an isomorphism class of representations and thus  $\mathbb{C}[\mathcal{V}]^{inv}$  is the coordinate ring of the corresponding moduli space.

One can construct elements of  $\mathbb{C}[\mathcal{V}]^{inv}$  by taking traces  $\varphi_{ii}(x)$  for different  $x \in \mathcal{A}$ , clearly the image would be invariant under the cyclic permutations of generators in each monomial and thus would depend

only on the element of the cyclic space  $A_{\natural} = \mathcal{A}/[\mathcal{A}, \mathcal{A}]$ . This induces a map  $\varphi_0 : \mathcal{A}/[\mathcal{A}, \mathcal{A}] \rightarrow \mathbb{C}[\mathcal{V}]^{inv}$  from  $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$  to the invariant subalgebra  $\mathbb{C}[\mathcal{V}]^{inv}$ . Denote the image of this map by  $\mathcal{H} := \varphi_0(\mathcal{A}/[\mathcal{A}, \mathcal{A}])$ .

**Lemma 4.** [Pro76]  $\mathbb{C}[\mathcal{V}]^{inv}$  is generated by  $\mathcal{H}$  as a commutative algebra.

**Example 5.** If  $A$  happens to be commutative, the representation functor  $\text{Rep}_1$  for  $N = 1$  will map it to itself. Moreover  $\mathbb{C}_{\mathcal{V}}^{inv}$  will coincide with  $\mathbb{C}_{\mathcal{V}}$  for this case.

**Example 6.** The simplest scheme here corresponds to the representations of  $A = \mathbb{C}\langle x^{(1)}, \dots, x^{(k)} \rangle$  — free algebra with  $k$  generators. In the absence of relations, the corresponding scheme is birational to  $\mathbb{C}^{kN^2}$ , a  $kN^2$ -dimensional vector space. And the corresponding sheaf of rational functions is nothing but the field of rational functions in  $kN^2$  variables.

**Example 7.** Another interesting special case corresponds to the so-called smooth algebras. Finitely generated algebra  $A$  is called smooth if  $\Omega^1 := \ker \mu = \{a_1 \otimes a_2 \mid a_1 a_2 = 0, a_1, a_2 \in A\}$  is projective as an inner bimodule. This guarantees that the representation scheme is actually a smooth affine variety. This case was in details studied in [CBEG07].

The major advantage of the Poisson formalism as compared to the Symplectic formalism is that it can be easily generalized beyond the smooth case.

**3.3. Bracket.** Define induced bracket  $\{, \}^{\mathcal{V}} : \mathbb{C}_{\mathcal{V}} \otimes \mathbb{C}_{\mathcal{V}} \rightarrow \mathbb{C}_{\mathcal{V}}$  on generators  $x_{ij}^{(m)}$  of  $\mathbb{C}[\mathcal{V}]$  by

$$(4a) \quad \{x_{ij}^{(m)}, x_{kl}^{(n)}\}^{\mathcal{V}} = \varphi \left( \{x^{(m)} \otimes x^{(n)}\}_{(kj), (il)} \right)$$

And then extend it to the entire  $\mathbb{C}_{\mathcal{V}}$  by Leibnitz identities w.r.t. both arguments. Namely, for all  $a, b, c \in \mathbb{C}_{\mathcal{V}}$ .

$$(4b) \quad \{ab, c\}^{\mathcal{V}} = a\{b, c\}^{\mathcal{V}} + b\{a, c\}^{\mathcal{V}},$$

$$(4c) \quad \{a, bc\}^{\mathcal{V}} = c\{a, b\}^{\mathcal{V}} + b\{a, c\}^{\mathcal{V}}.$$

As opposed to [dB08], bracket (4) in the context of Definition 1 is not necessary skew-symmetric and thus is not yet a Poisson bracket on  $\mathbb{C}_{\mathcal{V}}$ . It is a famous result of W. Crawley-Boevey [CB11] that any  $H_0$ -Poisson structure induces a conventional Poisson bracket on the moduli space of representations. In addition to that, we show in Proposition 12 that it comes with a Lie module action on the coordinate space of representations. In the case of bracket induced by the modified double Poisson bracket both are nothing but restrictions of (4) to  $\mathbb{C}_{\mathcal{V}}^{inv} \otimes \mathbb{C}_{\mathcal{V}}^{inv}$  and  $\mathbb{C}_{\mathcal{V}}^{inv} \otimes \mathbb{C}_{\mathcal{V}}$  respectively.

Easy to check that the above extension (4b) – (4c) is consistent with the double Leibnitz identity and relations  $\varphi(\mathcal{R})$  in the coordinate ring  $\mathbb{C}[\mathcal{V}]$ , namely

**Lemma 8.** Equations (4) define a unique linear map  $\{, \}^{\mathcal{V}} : \mathbb{C}[\mathcal{V}] \otimes \mathbb{C}[\mathcal{V}] \rightarrow \mathbb{C}[\mathcal{V}]$  given by

$$(5) \quad \forall x, y \in A : \{\varphi(x)_{ij}, \varphi(y)_{kl}\}^{\mathcal{V}} = \varphi(\{x, y\}'_{kj} \varphi(\{x, y\}''_{il}))$$

*Proof.* Define  $X = \varphi(x)$  and  $Y = \varphi(y)$ . In what follows assume the summation over repeating indexes

$$\{X_{ij}, Y_{kl} Z_{lm}\}^{\mathcal{V}} = \varphi(\{x, yz\}'_{kj} \varphi(\{x, yz\}''_{il}))_{im}.$$

On the other hand

$$\begin{aligned} \{x, yz\} &= (y \otimes 1)\{x, z\} + \{x, y\}(1 \otimes z) \\ &= y\{x, z\}' \otimes \{x, z\}'' + \{x, y\}' \otimes \{x, y\}'' z \end{aligned}$$

Which leads us to

$$\begin{aligned} \{X_{ij}, Y_{kl} Z_{lm}\}^{\mathcal{V}} &= Y_{kl} \varphi(\{x, z\}'_{lj} \varphi(\{x, z\}''_{im})) + \varphi(\{x, y\}'_{kj} \varphi(\{x, y\}''_{il})) Z_{lm} \\ &= Y_{kl} \{X_{ij}, Z_{lm}\}^{\mathcal{V}} + \{X_{ij}, Y_{kl}\}^{\mathcal{V}} Z_{lm} \end{aligned}$$

By the same derivation

$$\{X_{il} Y_{lj}, Z_{km}\}^{\mathcal{V}} = X_{il} \{Y_{lj}, Z_{km}\}^{\mathcal{V}} + Y_{lj} \{X_{il}, Z_{km}\}^{\mathcal{V}}.$$

Now, using the fact that  $\mathcal{A}$  is finitely generated by  $x^{(m)}$  we conclude that (4b) and (4c) uniquely extend  $\{ \_, \_ \}^\mathcal{V}$  on pairs of monomials. Moreover, defining ideal  $\varphi(\mathcal{R}) \subset \mathbb{C}[\mathcal{V}]$  for the coordinate ring  $\mathbb{C}[\mathcal{V}]$  is thus within the left and right kernel of  $\{ \_, \_ \}^\mathcal{V}$ . So  $\{ \_, \_ \}^\mathcal{V}$  extends uniquely to  $\mathbb{C}[\mathcal{V}]$ .  $\square$

Equation (5) immediately implies

**Corollary 9.** *For all  $a, b \in A$ ,  $\{\varphi_0(a), \varphi(b)\}^\mathcal{V} = \varphi(\{a, b\})$ .*

*Proof.*

$$(6) \quad \begin{aligned} \forall x, y \in A : \{\varphi(x)_{ii}, \varphi(y)_{kl}\}^\mathcal{V} &= \varphi(\{x, y\}'_{ki} \varphi(\{x, y\}''_{il}) = \varphi(\{x, y\}' \{x, y\}''_{kl}) = \varphi(\mu(\{x, y\}))_{kl} = \\ &= \varphi(\{x, y\}^\mathcal{V})_{kl}. \end{aligned}$$

$\square$

**Lemma 10.** *Equations (4) define a unique linear map  $\{ \_, \_ \}^\mathcal{V} : \mathbb{C}_\mathcal{V} \otimes \mathbb{C}_\mathcal{V} \rightarrow \mathbb{C}_\mathcal{V}$ .*

*Proof.* Taking into account Lemma 8 it would be enough to prove that the bracket  $\{ \_, \_ \}^\mathcal{V}$  can be extended to a properly localized ring. Let  $R$  be a  $\mathbb{C}$ -algebra s.t.  $\{ \_, \_ \}^\mathcal{V} : R \otimes R \rightarrow R$  is well defined and satisfies (4b) – (4c). For any multiplicative subset  $S \subset R$  and  $a \in S$ ,  $b \in R$  we immediately get  $\{a^{-1}, b\}^\mathcal{V} = -a^{-2}\{a, b\}^\mathcal{V}$  and  $\{b, a^{-1}\}^\mathcal{V} = -a^{-2}\{b, a\}^\mathcal{V}$ . This provides a unique extension of  $\{ \_, \_ \}^\mathcal{V}$  to  $S^{-1}R$ .

Thus for each distinguished open subset  $\mathcal{V}_f \subset \mathcal{V}$  we have a unique extension of  $\{ \_, \_ \}^\mathcal{V}$  to  $\Gamma(\mathcal{V}_f, \mathcal{O}_\mathcal{V})$ . Now denote by  $S(\mathcal{V}_f) \subset \Gamma(X_f, \mathcal{O}_\mathcal{V})$  the set of functions which are not a zero divisor on any stalk, we have a unique extension of  $\{ \_, \_ \}^\mathcal{V}$  to  $S(\mathcal{V}_f)^{-1}\Gamma(\mathcal{V}_f, \mathcal{O}_\mathcal{V}) = \Gamma(\mathcal{V}_f, \mathbb{C}_\mathcal{V})$ .  $\square$

**Lemma 11.** *Following restriction of  $\{ \_, \_ \}^\mathcal{V}$*

$$\{ \_, \_ \}^{inv} : \mathbb{C}_\mathcal{V}^{inv} \otimes \mathbb{C}_\mathcal{V}^{inv} \rightarrow \mathbb{C}_\mathcal{V}^{inv}$$

*is skew-symmetric, namely for all  $f, g \in \mathbb{C}_\mathcal{V}^{inv}$  we have  $\{f, g\}^\mathcal{V} \in \mathbb{C}_\mathcal{V}^{inv}$  and  $\{f, g\}^\mathcal{V} = -\{g, f\}^\mathcal{V}$ .*

*Proof.* In light of Leibnitz identities (4b) and (4c) it would be enough for us to show the statement for generators of  $\mathbb{C}[\mathcal{V}]^{inv}$ . So w.l.o.g we can assume that  $f, g \in \mathcal{H}$  (see Lemma 4). Under this assumption there exist  $x, y \in A$  s.t.  $f = \varphi_0(x)$  and  $g = \varphi_0(y)$ . Denote  $X := \varphi(x)$ ,  $Y := \varphi(y)$ . Using (6) we get

$$\{f, g\}^\mathcal{V} = \{X_{ii}, Y_{kk}\}^\mathcal{V} = \varphi(\{x, y\})_{kk} = \varphi_0(\{x, y\}) \in \mathbb{C}[\mathcal{V}]^{inv}$$

as a result

$$\{f, g\}^\mathcal{V} + \{g, f\}^\mathcal{V} = \varphi_0(\{x, y\} + \{y, x\}) = 0.$$

$\square$

**Proposition 12.** *The following restriction*

$$(7) \quad \{ \_, \_ \}^\mathcal{V} : \mathbb{C}_\mathcal{V}^{inv} \otimes \mathbb{C}_\mathcal{V} \rightarrow \mathbb{C}_\mathcal{V}$$

*satisfies the Jacobi identity for the left Loday bracket, for all  $f, g \in \mathbb{C}_\mathcal{V}^{inv}$  and  $h \in \mathbb{C}_\mathcal{V}$  :*

$$\{f, \{g, h\}^\mathcal{V}\}^\mathcal{V} - \{g, \{f, h\}^\mathcal{V}\}^\mathcal{V} = \{\{f, g\}^\mathcal{V}, h\}^\mathcal{V}.$$

*Proof.* For  $f, g \in \mathcal{H}$  and  $h \in \mathbb{C}[\mathcal{V}]$  the statement is a straightforward consequence of Corollary 9 and the fact that  $\{ \_, \_ \} : \mathcal{A}_\hbar \otimes \mathcal{A} \rightarrow \mathcal{A}$  is a Loday bracket. Denote

$$\phi(f, g, h) := \{f, \{g, h\}^\mathcal{V}\}^\mathcal{V} - \{g, \{f, h\}^\mathcal{V}\}^\mathcal{V} - \{\{f, g\}^\mathcal{V}, h\}^\mathcal{V}.$$

Since  $\phi$  is a derivation in its' last argument, left Jacobi identity extends for  $h \in \mathbb{C}_\mathcal{V}$ . Next, we have

$$\begin{aligned} \phi(f_1 f_2, g, h) - f_1 \phi(f_2, g, h) - f_2 \phi(f_1, g, h) &= \\ &= -\{g, f_1\}^\mathcal{V} \{f_2, h\}^\mathcal{V} - \{g, f_2\}^\mathcal{V} \{f_1, h\}^\mathcal{V} - \{f_2, g\}^\mathcal{V} \{f_1, h\}^\mathcal{V} - \{f_2, h\}^\mathcal{V} \{f_1, g\}^\mathcal{V} \end{aligned}$$

Note, that  $\{ \_, \_ \}^\mathcal{V}$  is not skew-symmetric in general, however by Lemma 11 we have for  $f_1, f_2, g \in \mathbb{C}_\mathcal{V}^{inv}$

$$\{g, f_1\}^\mathcal{V} + \{f_1, g\}^\mathcal{V} = 0, \quad \{f_2, g\}^\mathcal{V} + \{g, f_2\}^\mathcal{V} = 0$$

which is enough to conclude that for all  $f_1, f_2, g \in \mathbb{C}_\mathcal{V}^{inv}$  and  $h \in \mathbb{C}_\mathcal{V}$

$$\phi(f_1 f_2, g, h) = f_1 \phi(f_2, g, h) + f_2 \phi(f_1, g, h).$$

We also get for all  $f, g \in \mathbb{C}_{\mathcal{V}}^{inv}$  and  $g \in \mathbb{C}_{\mathcal{V}}$  s.t.  $f^{-1} \in \mathbb{C}_{\mathcal{V}}^{inv}$

$$\phi(f^{-1}, g, h) = -f^{-2}\phi(f, g, h).$$

Similar reasoning applies for the second argument which finalizes the proof.  $\square$

**Remark 13.** Proposition 12 defines a representation analogue of an  $H_0$ -Poisson structure. Note the dual properties, once  $H_0$ -Poisson structure factors through  $\{, \} : \mathcal{A}/[\mathcal{A}, \mathcal{A}] \otimes \mathcal{A} \rightarrow \mathcal{A}$ , the induced bracket defined above has to be restricted on invariant subalgebra  $\{, \}^{\mathcal{V}} : \mathbb{C}_{\mathcal{V}}^{inv} \otimes \mathbb{C}_{\mathcal{V}} \rightarrow \mathbb{C}_{\mathcal{V}}$  in order to satisfy Jacobi identity. Following ideas of [Tur14] we formulate this duality fundamentally in Section 6 when we show that one can generalize Proposition 12 beyond matrix representations.

**Corollary 14.** The following restriction

$$(8) \quad \{ -, - \}^{inv} : \mathbb{C}_{\mathcal{V}}^{inv} \otimes \mathbb{C}_{\mathcal{V}}^{inv} \rightarrow \mathbb{C}_{\mathcal{V}}^{inv}$$

is a Poisson bracket.

*Proof.* This statement follows from Corollary 2 and results of [CB11], however below we will present a direct proof using Proposition 12.

Indeed, by Lemma 11 this restriction is skew-symmetric, it satisfies Leibnitz identity in both arguments by definition (4b) – (4c). As a particular case of Proposition 12 it also satisfies Jacobi identity.  $\square$

**3.4. Casimir elements.** Action of the Poisson bracket (8) on the full representation scheme (7) provides a convenient way to construct Casimir elements. Recall

**Definition 15.** Element  $c \in \mathcal{A}$  is a right Casimir of bracket  $\{, \}$  if for all  $h \in \mathcal{A}/[\mathcal{A}, \mathcal{A}]$  we have  $\{h, c\} = 0$ .

**Remark 16.** It is worth noting that right Casimir elements are not necessary within the left kernel of the bracket beyond skew-symmetric case. For a particular example see (9) and (11).

Since  $\{, \}$  is a derivation in the second argument, the set of all right Casimir elements forms a subalgebra  $\mathcal{C} \subset \mathcal{A}$ . This subalgebra allows one to construct Casimir elements of the bracket on representation scheme.

**Proposition 17.** Assume that  $c \in \mathcal{A}$  is a right Casimir of bracket  $\{, \} : \mathcal{A}/[\mathcal{A}, \mathcal{A}] \otimes \mathcal{A} \rightarrow \mathcal{A}$ . Subalgebra  $\mathbb{C}[\varphi(c)_{kl}]$  generated by components of the matrix  $\varphi(c)$  consist of right Casimirs of the bracket  $\{, \}^{\mathcal{V}}$

*Proof.* For all  $h \in \mathcal{H}$  we have

$$\{\varphi_0(h), \varphi(c)_{kl}\}^{\mathcal{V}} = \{\varphi(h)_{ii}, \varphi(c)_{kl}\}^{\mathcal{V}} = \varphi(\{h, c\})_{kl} = 0.$$

Since  $\{, \}^{\mathcal{V}}$  satisfies Leibnitz identity w.r.t to both arguments we conclude that for all  $f \in \mathbb{C}_{\mathcal{V}}^{inv}$  and  $x \in \mathbb{C}[\varphi(c)_{kl}]$

$$\{f, x\}^{\mathcal{V}} = 0.$$

$\square$

Note, that in Proposition 17 it is essential to consider a restricted bracket  $\{ -, - \}^{\mathcal{V}}$  on  $\mathbb{C}_{\mathcal{V}}^{inv} \otimes \mathbb{C}_{\mathcal{V}}$  as defined in (17). If instead of element  $\mathbb{C}_{\mathcal{V}}^{inv}$  as a first argument we take arbitrary  $f \in \mathbb{C}_{\mathcal{V}}$  the bracket with a Casimir elements do not have to be zero.

**Corollary 18.** Let  $\mathcal{C} \subset \mathcal{A}$  be a subalgebra of right Casimir elements of bracket  $\{, \}$ , then  $\varphi_0(\mathcal{C}) \subset \mathbb{C}_{\mathcal{V}}^{inv}$  consist of Casimir elements of bracket  $\{, \}^{inv}$ .

This Corollary is especially useful when  $\mathcal{C}$  is finitely generated. We illustrate this method in Section 4.1.1.

## 4. EXAMPLES OF THE MODIFIED DOUBLE POISSON BRACKETS

**4.1. Bracket for Kontsevich system.** Here we describe a particular example of modified double bracket on  $\mathbb{C}\langle u^{\pm}, v^{\pm} \rangle$  introduced in [Art15]. This double bracket is not skew-symmetric and thus provides an example beyond the case considered in [dB08].

Let  $\mathcal{A}^+ = \mathbb{C}\langle u, v \rangle$  be a free associative algebra with two generators. Define a biderivation of  $\mathcal{A}$  on the generators as

$$(9) \quad \{u, v\}_K = -vu \otimes 1, \quad \{v, u\}_K = uv \otimes 1, \quad \{u, u\}_K = \{v, v\}_K = 0.$$

**Proposition 19.** [Art15] Biderivation  $\{ -, - \}_K$  is a modified double Poisson bracket.

Under the representation functor  $\text{Rep}_N$  our algebra is mapped to the commutative algebra  $\mathcal{A}_N = \mathbb{C}(u_{i,j}, v_{i,j})$  of rational functions in  $2N^2$  variables  $\{u_{i,j}, v_{i,j} \mid 1 \leq i, j \leq N\}$ . The corresponding affine scheme  $\mathcal{V}$  is just a  $2N^2$ -dimensional vector space over  $\mathbb{C}$ .

The induced bracket is a biderivation  $\{\cdot, \cdot\}^\mathcal{V} : \mathcal{A}_N \otimes \mathcal{A}_N \rightarrow \mathcal{A}_N$  defined on generators as

$$(10) \quad \begin{aligned} \{u_{ij}, v_{kl}\}^\mathcal{V} &= -\delta_{il} \sum_m v_{km} u_{mj} \\ \{v_{kl}, u_{ij}\}^\mathcal{V} &= \delta_{kj} \sum_m u_{im} v_{ml}. \end{aligned}$$

Proposition 14 implies that restriction of  $\{\cdot, \cdot\}^\mathcal{V}$  on invariant rational functions  $\{\cdot, \cdot\}^{inv} : \mathcal{A}_N^{inv} \otimes \mathcal{A}_N^{inv} \rightarrow \mathcal{A}_N^{inv}$  is a Poisson bracket.

4.1.1. *Dimensions of the symplectic leaves.* Element

$$(11) \quad c = uvu^{-1}v^{-1}$$

is a right Casimir of the  $H_0$ -Poisson bracket induced by (9) (see [Art15] for a proof). One can show that  $\text{Tr } \varphi(c^k)$  provide Casimirs for the induced bracket  $\{\cdot, \cdot\}^{inv}$ . The Poisson bracket on  $\mathbb{C}_\mathcal{V}^{inv}$  we defined earlier is degenerate due to existence of Casimirs. Which means that the Poisson tensor is not invertible at a generic point. In order to make it invertible (and thus induce a symplectic structure) one has to restrict the bracket to the subvariety corresponding to the fixed level of all Casimir functions (See e.g. [Arn78]). The codimension of such variety is, of course, simply the number of algebraically independent Casimir functions.

Based on direct computation of dimensions of symplectic leaves for bracket  $\{\cdot, \cdot\}^{inv}$  we come to the following

**Conjecture 20.** *There are exactly  $N - 1$  algebraically independent Casimir elements given by  $\text{Tr } \varphi(c^k)$  for the bracket  $\{\cdot, \cdot\}^{inv}$ .*

We summarize a computational evidence in favour of this conjecture in the Table 1. Here  $\dim L$  — dimension of a generic symplectic leaf,  $\text{codim } \varphi_0(c^k)$  — number of algebraically independent Casimirs provided by  $\varphi_0(c^k)$ .

$N$	$\dim \mathbb{C}_\mathcal{V}^{inv}$	$\dim L$	$\text{codim } \varphi_0(c^k)$
1	2	2	0
2	5	4	1
3	10	8	2
4	17	14	3
5	26	22	4
6	37	32	5

TABLE 1. Summary on tests of dimensions of symplectic leaf

4.2. **Other examples Double Poisson Brackets.** Below we present a couple of other examples of modified double Poisson brackets on  $\text{Free}_3 = \mathbb{C}\langle x_1, x_2, x_3 \rangle$ . Unlike (9), examples presented in this subsection are conjectural although very well tested. More examples and partial classification are in progress.

$$(12) \quad \begin{aligned} \{x_1, x_2\}^I &= -x_2 x_1 \otimes 1, & \{x_2, x_1\}^I &= x_1 x_2 \otimes 1, \\ \{x_2, x_3\}^I &= -x_2 \otimes x_3, & \{x_3, x_2\}^I &= x_2 \otimes x_3, \\ \{x_3, x_1\}^I &= -1 \otimes x_3 x_1, & \{x_1, x_3\}^I &= 1 \otimes x_1 x_3. \end{aligned}$$

Here all omitted brackets of generators are assumed to be zero.

$$(13) \quad \begin{aligned} \{x_1, x_2\}^{II} &= -x_1 \otimes x_2, & \{x_2, x_1\}^{II} &= x_1 \otimes x_2, \\ \{x_2, x_3\}^{II} &= x_3 \otimes x_2, & \{x_3, x_2\}^{II} &= -x_3 \otimes x_2, \\ \{x_3, x_1\}^{II} &= x_1 \otimes x_3 - x_3 \otimes x_1. \end{aligned}$$

**Conjecture 21.** Brackets (12) and (13) are modified double Poisson brackets on  $\text{Free}_3$ . Namely, corresponding biderivations satisfy (1c) and (1d).

We have tested equations (1c) and (1d) for all monomials up to length 5.

## 5. MODIFIED DOUBLE POISSON BI-VECTORS

Following [CB99, CBEG07, dB08] we introduce the notions of noncommutative vector fields and algebra of poly-vector fields. We slightly modify homomorphism from noncommutative poly-vectors to derivations introduced in [dB08] to include poly-derivations with no cyclic invariance.

**Definition 22.** For a finitely generated associative algebra  $\mathcal{A}$ , let  $\delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$  be a linear map satisfying the following form of Leibnitz identity

$$\delta(ab) = (a \otimes 1) \delta(b) + \delta(a) (1 \otimes b).$$

Then we call  $\delta$  a **noncommutative vector field**. The space of all noncommutative vector fields for a given algebra  $\mathcal{A}$  we denote by  $\mathcal{D}_{\mathcal{A}}$  in what follows.

In other words  $\delta \in \mathcal{D}_{\mathcal{A}}$  is a derivation of  $\mathcal{A}$  with a codomain  $\mathcal{A} \otimes \mathcal{A}$  treated as an outer  $\mathcal{A}$ -bimodule. The remaining structure of inner bimodule makes  $\delta$  itself an  $\mathcal{A}$ -bimodule, where the left-right action is defined s.t. for  $\delta \in \mathcal{D}_{\mathcal{A}}$  and for all  $a_1, a_2, x \in \mathcal{A}$ :

$$(14) \quad a_1 \delta(x) a_2 := (1 \otimes a_1) \delta(x) (a_2 \otimes 1) = \delta(x)' a_2 \otimes a_2 \delta(x)'.$$

Let  $\mathcal{DA} = T_{\mathcal{A}} \mathcal{D}_{\mathcal{A}}$  be the tensor algebra over  $\mathcal{A}$  generated by  $\mathcal{A}$ -bimodule (14). As suggested in [dB08] we call elements of  $\mathcal{DA}$  noncommutative poly-vector fields. One can equip  $\mathcal{DA}$  with a grading by assigning  $\deg a = 0$  for each  $a \in \mathcal{A}$  and  $\deg \delta = 1$  for each  $\delta \in \mathcal{D}_{\mathcal{A}}$ . We denote multiplication in  $\mathcal{DA}$  by  $\star$  and  $k^{th}$  homogenous component by  $(\mathcal{DA})_k$ . In Sweedler notations multiplication of noncommutative poly-vector fields becomes explicit

$$(15) \quad \begin{aligned} D &= \delta_1 \star \delta_2 \star \cdots \star \delta_k \in (\mathcal{DA})_k, \quad D : \mathcal{A}^{\otimes k} \rightarrow \mathcal{A}^{\otimes(k+1)}, \\ D(a_1 \otimes \cdots \otimes a_k) &= \delta_k(a_k)' \otimes \delta_{k-1}(a_{k-1})' \delta_k(a_k)'' \otimes \cdots \otimes \delta_1(a_1)' \delta_2(a_2)'' \otimes \delta_1(a_1)'' \end{aligned}$$

Unpaired components of vector fields in (15) provide a structure on an  $\mathcal{A}$ -bimodule, for each  $b \in \mathcal{A}$  and  $D \in (\mathcal{DA})_k$

$$\begin{aligned} (D \star b)(a_1 \otimes a_k) &= \delta_k(a_k)' b \otimes \cdots \otimes \delta_1(a_1)'', \\ (b \star D)(a_1 \otimes a_k) &= \delta_k(a_k)' \otimes \cdots \otimes b \delta_1(a_1)''. \end{aligned}$$

For each  $\mathcal{A}$ -bimodule one can define a notion of partial trace, so we introduce

**Definition 23.** Let  $P \in \mathcal{DA}$ , we call “partial trace over  $\mathcal{A}$ ” and denote as  $\text{tr}_{\mathcal{A}} P$  the following equivalence class

$$\text{tr}_{\mathcal{A}} P = P + [\mathcal{DA}, \mathcal{A}], \quad \text{where} \quad [\mathcal{DA}, \mathcal{A}] = \text{Span}\{Q \star a - a \star Q \mid a \in \mathcal{A}, Q \in \mathcal{DA}\},$$

or equivalently

$$\text{tr}_{\mathcal{A}} : \mathcal{DA} \rightarrow \mathcal{DA}/[\mathcal{DA}, \mathcal{A}].$$

**Definition 24.** Let  $\mathcal{D}_{\mathcal{A}^{\otimes k}}$  be the following space of  $k$ -derivations  $\delta : \mathcal{A}^{\otimes k} \rightarrow \mathcal{A}^{\otimes k}$  s.t. for all  $a_1, \dots, a_k \in \mathcal{A}$  and for all  $i \in \{1, \dots, k\}$

$$(16) \quad \begin{aligned} \delta(a_1 \otimes \cdots \otimes \underset{\uparrow i}{bc} \otimes \cdots \otimes a_k) &= \underbrace{(1 \otimes \cdots \otimes 1)}_{k-i} \otimes b \otimes \underbrace{(1 \otimes \cdots \otimes 1)}_i \delta(a_1 \otimes \cdots \otimes \underset{\uparrow i}{c} \otimes \cdots \otimes a_k) + \\ &+ \delta(a_1 \otimes \cdots \otimes \underset{\uparrow i}{b} \otimes \cdots \otimes a_k) \underbrace{(1 \otimes \cdots \otimes 1)}_{(1-i) \bmod k} \otimes c \otimes \underbrace{(1 \otimes \cdots \otimes 1)}_{((i-2) \bmod k)+1}. \end{aligned}$$

**Proposition 25.** Partial trace provides a linear homomorphism  $(\mathcal{DA}/[\mathcal{DA}, \mathcal{A}])_k \xrightarrow{\text{tr}_{\mathcal{A}}} \mathcal{D}_{\mathcal{A}^{\otimes k}}$  given by

$$(17) \quad (\text{tr}_{\mathcal{A}}(\delta_1 \star \cdots \star \delta_k))(a_1 \otimes \cdots \otimes a_k) = \delta_k(a_k)' \delta_1(a_1)'' \otimes \delta_{k-1}(a_{k-1})' \delta_k(a_k)'' \otimes \cdots \otimes \delta_1(a_1)' \delta_2(a_2)'.$$

*Proof.* First, recall that  $\delta_i(bc) = b \delta_i(c)' \otimes \delta_i(c)'' + \delta_i(b)' \otimes \delta_i(b)'' c$ , then substituting it in to the l.h.s. of (16) we get the desired result.  $\square$

**Corollary 26.** Let  $\delta_1, \delta_2 \in D_A$  be noncommutative vector fields, then  $R = \text{tr}_A(\delta_1 \star \delta_2)$  is a biderivation:

$$\begin{aligned} R(ab \otimes c) &= (1 \otimes a)(R(b \otimes c)) + (R(a \otimes c))(b \otimes 1), \\ R(a \otimes bc) &= (b \otimes 1)(R(a \otimes c)) + (R(a \otimes b))(1 \otimes c), \end{aligned}$$

where

$$R(a \otimes b) := \text{tr}_A(\delta_1 \star \delta_2)(a \otimes b) = \delta_2(b)' \delta_1''(a) \otimes \delta_1(a)' \delta_2(b)'.$$

**Definition 27.** Let  $P = \sum_i \delta_1^i \star \delta_2^i \in (\mathcal{DA})_2$  be a noncommutative bi-vector, we call  $P$  a **Modified double Poisson bi-vector** if the induced bi-derivation  $\text{tr}_A(\sum_i \delta_1^i \star \delta_2^i) : A \otimes A \rightarrow A \otimes A$  is a modified double Poisson bracket.

Proposition 25 should be compared to Proposition 4.1.1 in [dB08]. It is worth noting that we have used partial trace  $\text{tr}_A : \mathcal{DA} \rightarrow \mathcal{DA}/[\mathcal{DA}, \mathcal{A}]$  to construct bi-derivations by the corresponding poly-vector fields as opposed to the abelianization  $\mathcal{DA} \rightarrow \mathcal{DA}/[\mathcal{DA}, \mathcal{DA}]$  used in [dB08]. This allows one to take into consideration poly-derivations with no cyclic “anti-equivariance”.

As we have shown in Sec. 4 there are examples of the modified double Poisson brackets given by a biderivation which is substantially non-skew-symmetric. Below we present non-skew-symmetric double Poisson bivector which induces bracket (9).

**Poisson bivector for bracket (9).** Let  $\mathcal{A}^+ = \mathbb{C}\langle u, v \rangle$  be a free associative algebra with two generators. Define noncommutative vector fields  $\delta_1, \delta_2, \tilde{\delta}_1, \tilde{\delta}_2 \in D_A$  by their action on generators

$$\begin{aligned} \delta_1(u) &= 1 \otimes u, & \delta_1(v) &= 1 \otimes v, \\ \delta_2(u) &= u \otimes 1, & \delta_2(v) &= 0, \\ \tilde{\delta}_1(u) &= u \otimes 1, & \tilde{\delta}_1(v) &= v \otimes 1, \\ \tilde{\delta}_2(u) &= 1 \otimes u, & \tilde{\delta}_2(v) &= 0. \end{aligned}$$

**Proposition 28.**

$$\{\cdot, \cdot\}_K := \text{tr}_{\mathcal{A}^+}(\delta_1 \star \delta_2 - \tilde{\delta}_2 \star \tilde{\delta}_1)$$

*Proof.* On generators of  $\mathcal{A}$  formula above coincide with an example of modified double Poisson bracket  $\{\cdot, \cdot\}_K$  defined in (9). From Corollary 26 we conclude that they coincide for the entire domain  $\mathcal{A} \otimes \mathcal{A}$ .  $\square$

Thus  $\delta_1 \star \delta_2 - \tilde{\delta}_2 \star \tilde{\delta}_1 \in (\mathcal{DA})_2$  provides an essential example of a modified double Poisson bivector.

## 6. BRACKETS ON REPRESENTATION ALGEBRAS

In [Tur14, MT15] G. Massuyeau and V. Turaev suggested that double Poisson brackets induce Poisson brackets on representation algebras. In particular, it was shown that for each associative algebra  $\mathcal{A}$  and coalgebra  $\mathcal{M}$  one can define a commutative associative algebra  $\mathcal{A}_{\mathcal{M}}$  satisfying the following universal property: for any commutative algebra  $\mathcal{B}$  consider  $\text{Hom}(\mathcal{M}, \mathcal{B})$  as an associative algebra with convolution product, then for each  $s : \mathcal{A} \rightarrow \text{Hom}(\mathcal{M}, \mathcal{B})$  there exists a unique  $r : \mathcal{A}_{\mathcal{M}} \rightarrow \mathcal{B}$  s.t. the following diagram is commutative in the category of associative algebras

$$\begin{array}{ccc} A & \longrightarrow & \text{Hom}(\mathcal{M}, \mathcal{A}_{\mathcal{M}}) \\ & \searrow s & \downarrow r \\ & & \text{Hom}(\mathcal{M}, \mathcal{B}). \end{array}$$

In this section we investigate brackets on representation algebras induced by the modified double Poisson brackets. We start by a very brief review of representation algebras in Section 6.1 followed by Section 6.2 in which we show that modified double bracket induces a Poisson bracket on the “trace” subalgebra of the representation algebra. The “trace” subalgebra acts (as a Lie algebra) on the entire representation algebra by derivations.



**6.1. Representation Algebras.** Throughout this section let  $\mathcal{A}$  be an associative algebra and  $\mathcal{M}$  be a coassociative coalgebra with comultiplication  $\Delta : \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{M}$  and counit  $\epsilon : \mathcal{M} \rightarrow \mathbb{C}$ . Since  $\Delta$  is coassociative, for each  $m > 1$  comultiplication induces a unique map  $\Delta^m : \mathcal{M} \rightarrow \mathcal{M}^{\otimes m}$ . It will be useful for us to employ the following notations:  $\Delta^m(\alpha) =: \alpha^1 \otimes \cdots \otimes \alpha^m$  whenever  $\alpha \in \mathcal{M}$ .

Assume further, that  $\mathcal{M}$  is equipped with:

- (1) Bilinear form  $\nu : \mathcal{M} \otimes \mathcal{M} \rightarrow \mathbb{C}$  s.t.  $\forall \alpha, \beta \in \mathcal{M}$ ,  $\nu(\alpha, \beta^2)\beta^1 \otimes \beta^3 = \nu(\beta, \alpha^2)\alpha^1 \otimes \alpha^3$ ;
- (2) A “trace” element  $\tau \in \mathcal{M}$ , s.t.  $\forall \alpha \in \mathcal{M}$ ,  $\nu(\tau, \alpha) = \epsilon(\alpha)$ .

Together with comultiplication a bilinear form induces a map

$$\bar{\nu} : \mathcal{M} \otimes \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{M}, \quad \bar{\nu}(\alpha, \beta) = \nu(\alpha, \beta^2)\beta^1 \otimes \beta^3.$$

We denote the image of this map in Sweedler notations as  $\bar{\nu}(\alpha, \beta) = \alpha_\beta \otimes \beta^\alpha$ .

The requirement for  $\nu$  formulated above is equivalent to the symmetry of  $\bar{\nu}$ , namely

$$\nu(\alpha, \beta^2)\beta^1 \otimes \beta^3 = \nu(\beta, \alpha^2)\alpha^1 \otimes \alpha^3 \iff \alpha_\beta \otimes \beta^\alpha = \beta_\alpha \otimes \alpha^\beta$$

**Definition 29.** [Tur14] Let  $\mathcal{A}_\mathcal{M}$  be a commutative algebra generated by all such elements  $a_\alpha$ ,  $a \in A$ ,  $\alpha \in \mathcal{M}$ , subject to the relations

$$(1) \quad \forall k \in \mathbb{C}, a, b \in A, \text{ and } \alpha, \beta \in \mathcal{M},$$

$$k(a_\alpha) = (ka)_\alpha = a_{k\alpha}, \quad (a+b)_\alpha = a_\alpha + b_\alpha, \quad a_{\alpha+\beta} = a_\alpha + a_\beta;$$

$$(2) \quad \forall a, b \in A \text{ and } \alpha \in \mathcal{M},$$

$$(ab)_\alpha = a_{\alpha^1}b_{\alpha^2}.$$

We call  $\mathcal{A}_\mathcal{M}$  a **representation algebra** of  $A$  in  $\mathcal{M}$ .

**Lemma 30.** [Tur14]

$$(18a) \quad \alpha_\beta \otimes (\beta^\alpha)^1 \otimes (\beta^\alpha)^2 = \alpha_{(\beta^1)} \otimes (\beta^1)^\alpha \otimes \beta^2,$$

$$(18b) \quad (\alpha_\beta)^1 \otimes (\alpha_\beta)^2 \otimes \beta^\alpha = \beta^1 \otimes \alpha_{(\beta^2)} \otimes (\beta^2)^\alpha.$$

## 6.2. Modified brackets on representation algebras.

**Lemma 31.** Modified double Poisson bracket  $\{\cdot, \cdot\}$  induces a biderivation on representation algebra  $\mathcal{A}_\mathcal{M}$

$$\{\cdot, \cdot\}^\mathcal{M} : \mathcal{A}_\mathcal{M} \otimes \mathcal{A}_\mathcal{M} \rightarrow \mathcal{A}_\mathcal{M} \quad \{a_\alpha, b_\beta\}^\mathcal{M} := \{a, b\}'_{\alpha_\beta} \{a, b\}''_{\beta^\alpha} = \nu(\alpha, \beta^2) \{a, b\}'_{\beta^1} \{a, b\}''_{\beta^2}$$

*Proof.* Define  $\{\cdot, \cdot\}^\mathcal{M}$  on generators of  $\mathcal{A}_\mathcal{M}$  as above and then extend to arbitrary pairs of monomials by Leibnitz identity. We have to show that defining relations of  $\mathcal{A}_\mathcal{M}$  are annihilated by  $\{\cdot, \cdot\}^\mathcal{M}$ . For all  $a, b, c \in A$  and  $\alpha, \beta \in \mathcal{M}$  we get

$$\begin{aligned} \{(ab)_\alpha, c_\beta\}^\mathcal{M} &= \{ab, c\}'_{\alpha_\beta} \{ab, c\}''_{\beta^\alpha} \\ &= \{b, c\}'_{\alpha_\beta} (a\{b, c\}'')_{\beta^\alpha} + (\{a, c\}'b)_{\alpha_\beta} \{a, c\}''_{\beta^\alpha} \\ &= \{b, c\}'_{\alpha_\beta} a_{(\beta^\alpha)^1} \{b, c\}''_{(\beta^\alpha)^2} + \{a, c\}'_{(\alpha_\beta)^1} b_{(\alpha_\beta)^2} \{a, c\}''_{\beta^\alpha} \\ (\text{by Lemma 30}) &= a_{\alpha^1} \{b, c\}'_{(\alpha^2)_\beta} \{b, c\}''_{\beta^{(\alpha^2)}} + b_{\alpha^2} \{a, c\}'_{(\alpha^1)_\beta} \{a, c\}''_{\beta^{(\alpha^1)}} \\ &= a_{\alpha^1} \{b_{\alpha^2}, c_\beta\}^\mathcal{M} + b_{\alpha^2} \{a_{\alpha^1}, c_\beta\}^\mathcal{M} \\ &= \{a_{\alpha^1}b_{\alpha^2}, c_\beta\}^\mathcal{M}. \end{aligned}$$

Similar computation shows that  $\{a_\alpha, (bc)_\beta\}^\mathcal{M} = \{a_\alpha, b_{\beta^1}c_{\beta^2}\}^\mathcal{M}$ . □

**Lemma 32.** For all  $\alpha \in \mathcal{M}$  and  $a, b \in A$ :

$$\{a_\tau, b_\alpha\}^\mathcal{M} = (\{a, b\})_\alpha.$$

*Proof.* First, we use the fact that  $\tau$  is conjugate to counit to show that for all  $\alpha \in \mathcal{M}$

$$(19) \quad \tau_\alpha \otimes \alpha^\tau := \nu(\tau, \alpha^2) \alpha^1 \otimes \alpha^2 = \epsilon(\alpha^2) \alpha^1 \otimes \alpha^3 = \alpha^1 \otimes \alpha^2.$$

Now,

$$\begin{aligned} \{a_\tau, b_\alpha\}^\mathcal{M} &= \{a, b\}'_{\tau_\alpha} \{a, b\}''_{\alpha^\tau} \\ &\stackrel{(19)}{=} \{a, b\}'_{\alpha^1} \{a, b\}''_{\alpha^2} = (\{a, b\}' \{a, b\}'')_\alpha \\ &= (\{a, b\})_\alpha. \end{aligned}$$

□

**Proposition 33.** *The following restriction*

$$\{, \}^\mathcal{M} : \mathcal{A}_\tau \otimes \mathcal{A}_\mathcal{M} \rightarrow \mathcal{A}_\mathcal{M}$$

*satisfies Jacobi identity for left Loday bracket, namely for all  $\alpha \in \mathcal{M}$  and  $a, b, c \in \mathcal{A}$ :*

$$\{a_\tau, \{b_\tau, c_\alpha\}^\mathcal{M}\}^\mathcal{M} - \{b_\tau, \{a_\tau, c_\alpha\}^\mathcal{M}\}^\mathcal{M} = \{\{a_\tau, b_\tau\}^\mathcal{M}, c_\alpha\}^\mathcal{M}.$$

*Proof.* Compose (2c) with Lemma 32

□

**Corollary 34.** *Subspace  $\mathcal{A}_\tau \subset \mathcal{A}_\mathcal{M}$  is a Lie algebra w.r.t.  $\{, \}^\mathcal{M}$ .*

**Corollary 35.** *Let  $\mathbb{C}[\mathcal{A}_\tau] \subset \mathcal{A}_\mathcal{M}$  be a commutative algebra generated by  $\mathcal{A}_\tau$ , then the following restriction of  $\{, \}^\mathcal{M}$*

$$\{, \}^\tau : \mathbb{C}[\mathcal{A}_\tau] \otimes \mathbb{C}[\mathcal{A}_\tau] \rightarrow \mathbb{C}[\mathcal{A}_\tau]$$

*is a Poisson bracket.*

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#### 7. CONCLUSION AND DISCUSSION

Throughout this note we have shown that one can generalize fundamental results of [dB08] with no assumption of skew-symmetry of the double bracket

$$\{a, b\}' \otimes \{a, b\}'' = -\{b, a\}'' \otimes \{b, a\}'.$$

We have presented an essential example of non-skew-symmetric modified double Poisson bracket and have constructed the corresponding modified double Poisson bi-vector.

We conclude by remark on Jacobi identity for the double Poisson bracket

**Jacobi identity beyond triple derivations.** Jacobi identity for the Loday bracket induced by the modified double bracket can be presented in the following form

$$(20) \quad 0 = \{H_1, \{H_2, x\}\} - \{H_2, \{H_1, x\}\} - \{\{H_1, H_2\}, x\} = \mu(D_1 + D_2),$$

where

$$\begin{aligned} D_1 &:= R_{12}R_{23} - R_{23}R_{13} - R_{13}R_{12}, \\ D_2 &:= \sigma_{12}(R_{13}R_{23} - R_{21}R_{13} - R_{23}R_{12}), \end{aligned}$$

$$R_{m,n} : \mathcal{A}^{\otimes k} \rightarrow \mathcal{A}^{\otimes k}, \quad R_{m,n}(a_1 \otimes \cdots \otimes a_k) = a_1 \otimes \cdots \otimes \underset{\uparrow i}{\{a_i, a_j\}'} \otimes \cdots \otimes \underset{\uparrow j}{\{a_i, a_j\}''} \otimes \cdots \otimes a_k.$$

Definition of the usual double Poisson bracket [dB08] included strong requirement of the skew-symmetry  $R_{m,n} = -\sigma_{(m,n)} R_{n,m} \sigma_{(m,n)}$  and the so-called double Jacobi identity (analogue of Yang-Baxter equation for double bracket). Together they guarantee that  $D_1$  and  $D_2$  are triple derivations and vanish separately. This fact was heavily used in various classification problems of double Poisson brackets [ORS13, Pow16].

However, this is no longer the case for modified double Poisson brackets (in particular, it fails for (9)). Indeed, consider a defect of  $D_1$  being a derivation w.r.t. to the first argument

$$(21) \quad D_1(a_1 a_2 \otimes b \otimes c) - (1 \otimes a_1 \otimes 1) D_1(a_2 \otimes b \otimes c) - D_1(a_1 \otimes b \otimes c)(a_2 \otimes 1 \otimes 1) = \\ -\{a_2, c\}' \otimes \{b, a_1\}' \otimes \{b, a_1\}'' \{a_2, c\}'' - \{a_2, c\}' \otimes \{a_1, b\}'' \otimes \{a_1, b\}' \{a_2, c\}''.$$

Beyond the skew-symmetric case, the right hand side of (21) doesn't vanish in general. Instead, only the combination composed with multiplication map (20) vanishes.

The latter makes classification problem for modified double Poisson brackets rather challenging. On the other hand, Definition 1 is completely explicit for finitely generated algebra and it is enough to define its' action on generators. It is an interesting topic of further research to formulate an explicit condition (namely, that can be tested on generators only) which can replace (1c) and (1d) in Definition 1.

## APPENDIX A. SPECTRAL CURVE FOR KONTSEVICH SYSTEM

In this appendix we describe an application of modified double Poisson brackets to the noncommutative Integrable System suggested in [Kon11]. We show that it induces Liouville Integrable System on the moduli space of representations of the underlying associative algebra for small  $N$ . Moreover, using Proposition 12 we show that the corresponding Hamilton flows extend to the derivations of the entire coordinate space of representations. Finally, from noncommutative Lax pair suggested in [EW12] we construct the corresponding spectral curve and compute its genus.

**A.1. Kontsevich system.** Let  $A = \mathbb{C}\langle u^{\pm 1}, v^{\pm 1} \rangle$  denote the group algebra of the free group with two generators. Consider derivation  $\frac{d}{dt} : A \rightarrow A$  defined by

$$(22) \quad \begin{cases} \frac{du}{dt} = uv - uv^{-1} - v^{-1}, \\ \frac{dv}{dt} = -vu + vu^{-1} + u^{-1}. \end{cases}$$

Recall the modified double Poisson bracket (9)

$$(23) \quad \{u, v\}_K = -vu \otimes 1, \quad \{v, u\}_K = uv \otimes 1, \quad \{u, u\}_K = \{v, v\}_K = 0.$$

Denote the induced  $H_0$ -Poisson structure as

$$(24) \quad \{, \}_K : A \otimes A \rightarrow A; \quad \forall a, b \in A, \{a, b\}_K = \mu(\{a, b\}_K).$$

Bracket (23) was suggested in [Art15] by the author to show the integrability of (22). We have the following list of properties:

- Derivation (22) is a generalized Hamilton flow w.r.t. bracket (24), namely

$$\forall x \in A, \quad \frac{dx}{dt} = \{h, x\}_K, \quad \text{where} \quad h = u + v + u^{-1} + v^{-1} + u^{-1}v^{-1}.$$

- There exists an infinite family of commuting flows, for all  $k, j \in \mathbb{N}$

$$\frac{d}{dt_k} : A \rightarrow A, \quad \frac{d}{dt_k}(x) := \{h^k, x\}; \quad \left[ \frac{d}{dt_k}, \frac{d}{dt_j} \right] = 0.$$

- Group commutator  $c = uvu^{-1}v^{-1}$  generates the subalgebra of right Casimirs of bracket (24)

$$\forall x \in A/[A, A], \quad \forall C \in \mathbb{C}[c] : \quad \{x, C\}_K = 0.$$

**A.2. Basic example. Matrix representations for  $N = 2$ .** In this subsection we show that Kontsevich system induces a conventional Integrable System in the Liouville sense on the moduli space of 2-dimensional representations. Moreover, we show that Hamilton flows extend to the entire coordinate space of representations.

Define an 8-dimensional manifold  $\mathcal{M} \subset R^8$  with coordinates  $u_{11}, u_{12}, u_{21}, u_{22}, v_{11}, v_{12}, v_{21}, v_{22}$  s.t.  $u_{11}u_{22} - u_{12}u_{21} \neq 0$  and  $v_{11}v_{22} - v_{12}v_{21} \neq 0$ . Let

$$(25) \quad \varphi(u) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \quad \varphi(v) = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}.$$

Then  $\varphi : \mathcal{M} \times A \rightarrow Mat(2, \mathbb{C})$  provides a representation of  $A$  for each given  $m \in \mathcal{M}$ .

Algebra  $\mathbb{C}[\mathcal{M}^{inv}]$  of all  $GL(2, \mathbb{C})$ -invariant functions on  $\mathcal{M}$  is generated by  $\varphi_0(\{u, v, u^2, uv, v^2\})$ .

$$\begin{aligned}
(26) \quad t_1 &:= \text{Tr } \varphi(u) = u_{11} + u_{22} \\
t_2 &:= \text{Tr } \varphi(v) = v_{11} + v_{22} \\
t_3 &:= \text{Tr } \varphi(uu) = u_{11}^2 + 2u_{12}u_{21} + u_{22}^2 \\
t_4 &:= \text{Tr } \varphi(vv) = u_{11}v_{11} + u_{21}v_{12} + u_{12}v_{21} + u_{22}v_{22} \\
t_5 &:= \text{Tr } \varphi(vv) = v_{11}^2 + 2v_{12}v_{21} + v_{22}^2
\end{aligned}$$

In terms of variables  $t$  defined in (26) we get the following brackets

$$\begin{aligned}
(27) \quad \{t_1, t_2\}^{inv} &= -t_4 \\
\{t_1, t_3\}^{inv} &= 0 \\
\{t_1, t_4\}^{inv} &= \frac{1}{2}(t_1^2 t_2 - t_2 t_3 - 2t_1 t_4) \\
\{t_1, t_5\}^{inv} &= t_1 t_2^2 - 2t_2 t_4 - t_1 t_5 \\
\{t_2, t_3\}^{inv} &= -t_1^2 t_2 + t_2 t_3 + 2t_1 t_4 \\
\{t_2, t_4\}^{inv} &= \frac{1}{2}(-t_1 t_2^2 + 2t_2 t_4 + t_1 t_5) \\
\{t_2, t_5\}^{inv} &= 0 \\
\{t_3, t_4\}^{inv} &= t_1^3 t_2 - t_1 t_2 t_3 - t_1^2 t_4 - t_3 t_4 \\
\{t_3, t_5\}^{inv} &= 2(t_1^2 t_2^2 - 2t_1 t_2 t_4 - t_3 t_5) \\
\{t_4, t_5\}^{inv} &= t_1 t_2^3 - t_2^2 t_4 - t_1 t_2 t_5 - t_4 t_5
\end{aligned}$$

The subalgebra of Casimir functions is generated by a single element  $\text{Tr } \varphi(c)$  and one can check independently that symplectic leaf of (27) has dimension 4 (see Table 1). The two Hamilton functions  $H_1 = \text{Tr } \varphi(h)$  and  $H_2 = \text{Tr } \varphi(h^2)$  are algebraically independent. Next,  $\{H_1, H_2\}^{inv} = 0$  by the fact that  $\{h^m, h^n\} \equiv 0 \pmod{[A, A]}$  and Corollary 9. Thus, system (22) induces an Integrable System in the Liouville sense on  $\mathcal{M}^{inv}$ .

By Proposition 12 the full coordinate space of representations  $\mathbb{C}[\mathcal{M}]$  is a Lie module of  $\mathbb{C}[\mathcal{M}^{inv}]$  with a Lie algebra structure on  $\mathbb{C}[\mathcal{M}^{inv}]$  given by the Poisson bracket (27). In particular, this implies that  $H_1$  and  $H_2$  generate commuting flows on the full  $\mathbb{C}[\mathcal{M}]$  which preserve  $\mathbb{C}[\mathcal{M}^{inv}]$ .

**A.3. Spectral Curve.** Although  $\mathbb{C}[h] \rightarrow A/[A, A]$  provides a subspace of hamiltonians in involution it doesn't span the maximal commuting Lie subalgebra w.r.t. to the Lie bracket induced by (24) (for example see eq. (26) on p. 1237 of [Art15]). In [EW12] O. Efimovskaya and T. Wolf suggested a noncommutative Lax pair with a spectral parameter for system (22) and conjectured that it will provide all "trace integrals", elements of  $A$  invariant under (22) modulo  $[A, A]$

$$\frac{d}{dt}L = [L, M],$$

where

$$(28) \quad L = \begin{pmatrix} v^{-1} + u & \lambda v + v^{-1}u^{-1} + u^{-1} + 1 \\ v^{-1} + \frac{1}{\lambda}u & v + v^{-1}u^{-1} + u^{-1} + \frac{1}{\lambda} \end{pmatrix}, \quad M = \begin{pmatrix} v^{-1} - v + u & \lambda v \\ v^{-1} & u \end{pmatrix}.$$

Denote the coefficients of the "noncommutative spectral curve" as  $\text{Tr } L(\lambda)^k =: \sum_{j=-k}^k H_{k,j} \lambda^j$ . We have checked that for all  $k, m \leq 5$  and arbitrary  $j, l$

$$\{H_{k,j}, H_{m,l}\}_K \equiv 0 \pmod{[A, A]}.$$

The latter combined with results of O. Efimovskaya and T. Wolf suggests the following

**Conjecture 36.** *Image of  $\text{Span}(H_{k,l})$  in  $A/[A, A]$  under natural projection is a maximal commutative Lie subalgebra of  $A/[A, A]$  w.r.t. to the Lie bracket induced by (24).*

For each  $N$  and each general point in  $\mathcal{A}_N$ , Lax matrix (28) gives rise to an algebraic curve in  $\lambda$  and  $\nu$

$$(29) \quad \det(\varphi(L(\lambda)) - \nu) = 0.$$

The coefficients of the curve belong to the subalgebra  $\mathbb{C}[\varphi_0(H_{k,l})] \subset \mathcal{A}_N^{inv}$  of invariant polynomials generated by  $\varphi_0(H_{k,l})$ . Assuming Conjecture 36, by Corollary 9 we get

**Corollary 37.** *Subalgebra  $\mathbb{C}[\varphi_0(H_{k,l})]$  generated by the coefficients of the spectral curve (29) is Poisson commutative with respect to the induced bracket  $\{\cdot, \cdot\}^{inv}$ .*

Below we present a Newton graphs of the curve (29) for small  $N$ . For each  $N$  the spectral curve appears to be highly singular, we have computed its arithmetic genus for general point in  $\mathcal{A}_N$  for  $N \leq 4$ . Table 2 suggests that the genus is equal to  $N^2$ .

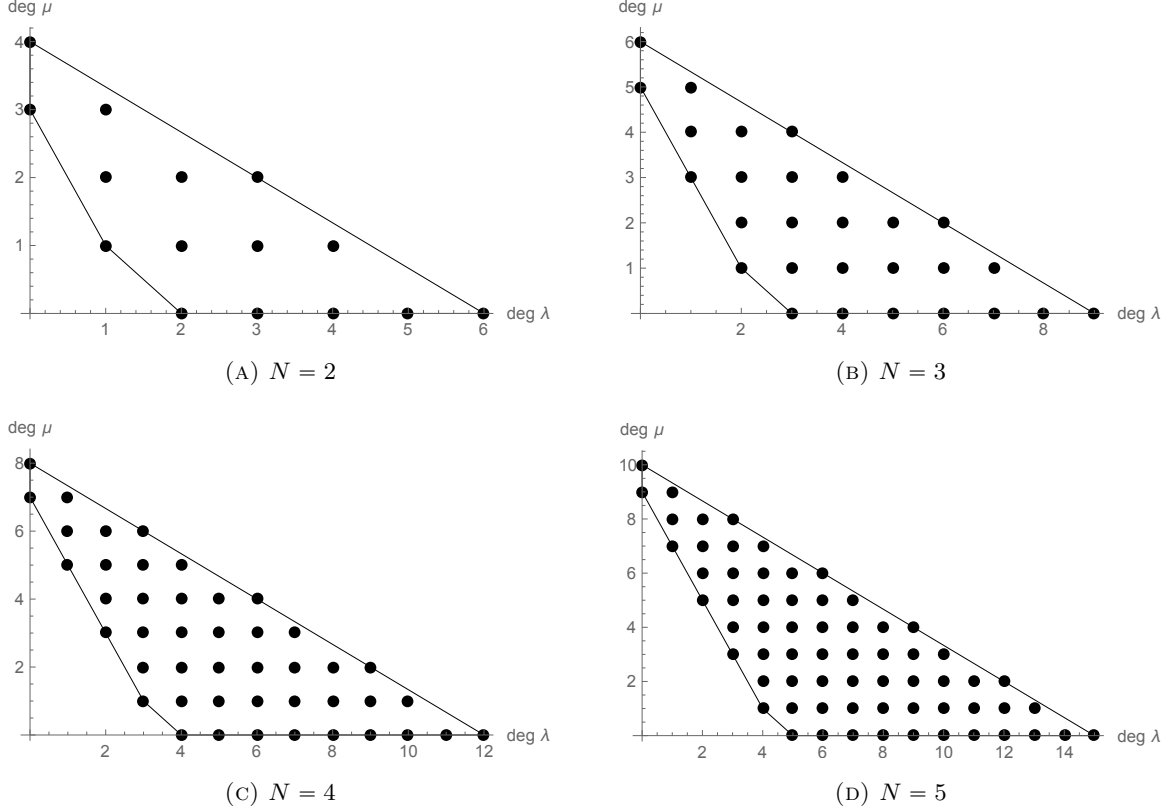


FIGURE 1. Newton graphs of the Spectral curve

$N$	genus	degree
2	4	6
3	9	9
4	16	12

TABLE 2. Genus of the Spectral curve

## REFERENCES

- [Arn78] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer-Verlag, New York and Berlin, 1978.
- [Art15] S. Arthamonov. Noncommutative inverse scattering method for the Kontsevich system. *Lett. Math. Phys.*, 105(9):1223–1251, 2015.
- [BT16] Claudio Bartocci and Alberto Tacchella. Poisson-nijenhuis structures on quiver path algebras. *arXiv preprint arXiv:1604.02012*, 2016.
- [CB99] William Crawley-Boevey. Preprojective algebras, differential operators and a conze embedding for deformations of kleinian singularities. *Commentarii Mathematici Helvetici*, 74(4):548–574, 1999.
- [CB11] William Crawley-Boevey. Poisson structures on moduli spaces of representations. *Journal of Algebra*, 325(1):205 – 215, 2011.

- [CBEG07] William Crawley-Boevey, Pavel Etingof, and Victor Ginzburg. Noncommutative geometry and quiver algebras. *Advances in Mathematics*, 209(1):274 – 336, 2007.
- [dB08] Michael Van den Bergh. Double Poisson algebras. *Trans. Amer. Math. Soc.*, 360:5711–5769, 2008.
- [EW12] Olga Efimovskaya and Thomas Wolf. On Integrability of the Kontsevich Non-Abelian ODE system. *Letters in Mathematical Physics*, 100(2):161–170, 2012.
- [Kon93] Maxim Kontsevich. Formal (non)-commutative symplectic geometry. In Israel M. Gelfand, Lawrence Corwin, and James Lepowsky, editors, *The Gelfand Mathematical Seminars, 1990–1992*, pages 173–187. Birkhauser Boston, 1993.
- [Kon11] Maxim Kontsevich. Noncommutative Identities. *arXiv:1109.2469*, 2011.
- [MT15] Gwenael Massuyeau and Vladimir Turaev. Brackets in representation algebras of hopf algebras. *arXiv:1508.07566*, 2015.
- [ORS12] A.V. Odesskii, V.N. Rubtsov, and V.V. Sokolov. Bi-Hamiltonian ordinary differential equations with matrix variables. *Theoretical and Mathematical Physics*, 171(1):442–447, 2012.
- [ORS13] A. Odesskii, V. Rubtsov, and V. Sokolov. Poisson brackets on free associative algebras. *Contemporary Mathematics*, 592:295, 2013.
- [Pow16] Geoffrey Powell. On double Poisson structures on commutative algebras. *arXiv preprint arXiv:1603.07553*, 2016.
- [Pro76] Claudio Procesi. The invariant theory of  $n \times n$  matrices. *Advances in Mathematics*, 19(3):306–381, 1976.
- [PVdW08] Anne Pichereau and Geert Van de Weyer. Double Poisson cohomology of path algebras of quivers. *Journal of Algebra*, 319(5):2166–2208, 2008.
- [Tur14] Vladimir Turaev. Poisson–gerstenhaber brackets in representation algebras. *Journal of Algebra*, 402:435–478, 2014.